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Analysis of the Dynamics of SI-SI-SEIR Avian Influenza A(H7N9) Epidemic Model with Re-infection

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Abstract

The spread of Avian influenza in Asia, Europe and Africa ever since its emergence in 2003, has been endemic in many countries. In this study, a non-linear SI-SI-SEIR Mathematical model with re-infection as a result of continuous contact with both infected poultry from farm and market is proposed. Local and global stability of the three equilibrium points are established and numerical simulations are used to validate the results.

1 Introduction

Avian Influenza is a virus majorly spread in birds but can also be transmitted to humans with the human infections acquired through direct contact with infected animals or contaminated environment. Human are infected with avian influenza

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Nomenclature

$S_h(t)$	Susceptible human		Λ_h	recruitment rates of human
$E_h(t)$	exposed human		Λ_f	recruitment rates of poultry
$I_h(t)$	infected human		μ_h	natural mortality rates of human
$R_h(t)$	recovered human		μ_a	natural mortality rates of poultry
$S_f(t)$	susceptible poultry in farm		μ_d	disease-related death rates of infected human
$I_f(t)$	infected poultry in farm		μ_f	disease-related death rates of infected poultry
$S_m(t)$	susceptible poultry in market		α_0	proportion of poultry from farms to markets
$I_m(t)$	infected poultry in market		r	recovery rate of infected human
ψ	re-infection rate		β_h	human transmission rate
α_h	human progression rate from the latent period of infection to the infected class		β_f	transmission coefficient from infective poultry of farms to susceptible poultry of farms
β_m	contact rate from infective poultry of markets to susceptible poultry of markets		β_h	transmission rate from infected poultry of markets and farm to susceptible human

virus subtypes A(H5N1), A(H7N9), and A(H7N9). In 2013, the disease was reported for the first time across China, with over 1500 human cases reported and many deaths. The period of incubation ranges from 1 to 10 days, with an average of 5 days. The signs and symptoms are upper respiratory infection (fever and cough) which may progress to severe pneumonia, acute respiratory distress syndrome and death. The novel Coronavirus disease 2019 (COVID-19) has similar disease presentation with the Influenza virus; they are both zoonotic. Both the influenza virus and coronavirus cause respiratory disease ranging from mild illness to severe disease and death. The two viruses are transmitted by contact or droplets. For this reason, they can both be prevented by proper hand hygiene and good respiratory etiquette [1, 2, 3].

Iwami et al. [4] proposed a mathematical model that considered two types of avian influenza outbreak which may occur if humans fail to stop the spread of the disease. Liu and Fang [5] developed a dynamical model of avian influenza A(H7N9) to check the effect of the spread between poultry and poultry, poultry and human, and human and human and it was established that the likelihood of human-to-human transmission of the avian influenza A(H7N9) is low. Though the probability is low, the possibility of human-to-human transmission should still be considered. Che et al. [6] furthered the research by including saturated contact rate in the model for a highly pathogenic avian influenza epidemic. Chen and Wen [7] diversified the model by treating a bilinear disease incidence case with mutant avian influenza A(H7N9) virus, meanwhile, the highly pathogenic avian influenza epidemic model in the presence of vertical transmission function in poultry was proposed in [8]. The model proposed and investigated in [9] describes the transmission dynamics of avian influenza A(H7N9) between human and poultry. The model in [9] was further extended in [10] by introducing the possibility of re-infection. The results of the research indicated that a recovered individual who continue to have contact with an infected poultry may be re-infected with the disease. More recently, the spread of the avian influenza has attracted the attention of many researchers who working vehemently to unravel the dynamics of the transmission and as a result, bring an end to the spread of the disease. Advancement in the study of avian influenza include the models with Vaccination and Seasonality, the study of the Antigenic Variant, effects of other illnesses on patients infected with Avian Influenza A (H7N9) Virus, spatiotemporal variation and hotspot detection of the Avian Influenza A(H7N9) Virus, specificity, kinetics and longevity of antibody responses to avian influenza A(H7N9) [11, 12, 13, 14, 15, 16]. The studies have been very successful so far.

It is important to note that despite the intense concentration on the dynamics of the avian influenza A(H7N9) virus, the SI-SI-SEIR model with re-infection has not been explored. In this paper, as a further advancement to [10], an SI-SI-SEIR model for the transmission of an avian influenza A(H7N9) virus with re-infection is proposed. The dynamics of the transmission is unraveled by the available mathematical tools. This paper is organized as follows: Section 2 presents the SI-SI-SEIR model with re-infection to study avian influenza A(H7N9) transmission, the reproduction number and the existence of the equilibria points are established in Section 3 and Section 4 shows that the equilibria are locally, and globally asymptotically stable using the Lyapunov functions. We present the numerical simulation to validate our results in Section 5 and give the conclusion in Section 6.

2 The Model

Considering the work of [10] and [9], the novelty of this work is included in the following four assumptions; 1) the natural and the disease-induced death rate are the same for the poultry population. 2) the human population is classified with the inclusion of Exposed class to the susceptible, infected and the recovered compartments. 3) every bird is moved to market at once, if the progression rate of susceptible poultry from farm to market is α_0, then the progression rate of infected poultry to market will be $(1-\alpha_0)$. 4) human get infected not only in market but also in farm. 5) Human re-infection with the disease is expressed as $\psi\beta_h\left(I_f+I_m\right)R_h$, where ψ is the re-infection rate, β_h is the human transmission rate.

We propose the epidemic dynamics model of avian influenza A(H7N9) virus as

$$S_f' = \Lambda_f - \beta_f S_f I_f - (\mu_a+\alpha_0)S_f, \tag{2.1}$$
$$I_f' = \beta_f S_f I_f - (1-\alpha_0)I_f - (\mu_a+\mu_f)I_f, \tag{2.2}$$
$$S_m' = \alpha_0 S_f - \beta_m S_m I_m - \mu_a S_m, \tag{2.3}$$
$$I_m' = \beta_m S_m I_m + (1-\alpha_0)I_f - (\mu_a+\mu_f)I_m, \tag{2.4}$$
$$S_h' = \Lambda_h - \beta_h(I_f+I_m)S_h - \mu_h S_h, \tag{2.5}$$
$$E_h' = \beta_h(I_f+I_m)(S_h+\psi R_h) - (\alpha_h+\mu_h)E_h, \tag{2.6}$$

$$I_h' = \alpha_h E_h - (r+\mu_d+\mu_h)I_h, \tag{2.7}$$
$$R_h' = rI_h - \beta_h(I_f+I_m)\psi R_h - \mu_h R_h. \tag{2.8}$$

The disease free equilibrium is obtained from Eq. (2.1-2.8) as

$$\left(S_f^0, I_f^0, S_m^0, I_m^0, S_h^0, E_h^0, I_h^0, R_h^0\right) = \left(\frac{\Lambda_f}{(\mu_a+\alpha_0)}, 0, \frac{\alpha_0\Lambda_f}{\mu_a(\mu_a+\alpha_0)}, 0, 0, 0, 0, 0\right). \tag{2.9}$$

The reproduction number $\Re_0$ is defined as the expected number of secondary cases produced in a totally sensitive population by a typical infective individual during infectious period at a disease free equilibrium. The effective reproduction number is used to ascertain the transmission ability of a disease. The reproduction number is affected by the rate of contacts in the host population, the probability of infection transmission during contact and the contagious duration, hence, we obtain the reproduction number using the next generation matrix proposed by [17]

$$f = \begin{pmatrix} \beta_f S_f I_f \\ \beta_m S_m I_m \\ \beta_h (I_f + I_m)(S_h + \psi R_h) \\ 0 \end{pmatrix}, \tag{2.10}$$

$$v = \begin{pmatrix} (1-\alpha_0) I_f + (\mu_a + \mu_f) I_f \\ (\mu_a + \mu_f) I_m - (1-\alpha_0) I_f \\ (\alpha_h + \mu_h) E_h \\ (r + \mu_d + \mu_h) I_h - \alpha_h E_h \end{pmatrix}. \tag{2.11}$$

Then

$$\Re_0 = \rho\left(F^0 V^{-1}\right) = \max\left(\Re_{01}, \Re_{02}\right), \tag{2.12}$$

where

$$\Re_{01} = \frac{\beta_f \Lambda_f}{(\alpha_0 + \mu_a)(1 - \alpha_0 + \mu_a + \mu_f)} \quad \text{and} \quad \Re_{02} = \frac{\alpha_0 \beta_m \Lambda_f}{\mu_a (\alpha_0 + \mu_a)(\mu_a + \mu_f)}.$$

3 Existence of Equilibria

We prove the theorem on the existence of equilibria.

Theorem 1. *For system (2.1-2.8),*

1. *whenever* $\Re_{01} > 1$, $\Re_{02} > 1$, *there is the unique endemic equilibrium*

$$U^{***} = \left(S_f^{***}, I_f^{***}, S_m^{***}, I_f^{***}, S_h^{***}, E_h^{***}, I_h^{***}, R_h^{***}\right).$$

2. *whenever* $\Re_{01} < 1$, $\Re_{02} > 1$, *there is the unique boundary equilibrium*

$$U^* = \left(S_f^*, 0, S_m^*, I_f^*, S_h^*, E_h^*, I_h^*, R_h^*\right).$$

3. *whenever* $\Re_{01} > 1$, $\Re_{02} < 1$, *another unique boundary equilibrium*

$$U^{**} = \left(S_f^{**}, I_f^{**}, S_m^{**}, 0, S_h^{**}, E_h^{**}, I_h^{**}, R_h^{**}\right).$$

Proof. The three conditions stated above are considered

1. Consider the first case where $I_f \neq 0$ and $I_m \neq 0$, then it is obtained from Eq.(2.2) and (2.1) respectively that

$$S_f^{***} = \frac{(1 - \alpha_0 + \mu_a + \mu_f)}{\beta_f},\ I_f^{***} = \frac{(\mu_a + \alpha_0)}{\beta_f}\left(\Re_{01} - 1\right). \tag{3.1}$$

Since all parameters are positive and $I_f^{***} > 0$, then we require $\Re_{01} > 1$. From Eq. (2.3),

$$S_m^{***} = \frac{\alpha_0 S_f^{***}}{\beta_m I_m^{**} + \mu_a}. \tag{3.2}$$

Substituting (3.2) into Eq.(2.4) gives

$$a I_m^{***2} + b I_m^{***} + c = 0, \tag{3.3}$$

where

$$a = -\beta_m\left(\mu_a + \mu_f\right),\ b = \alpha_0 \beta_m S_f^{***} + \left(1 - \alpha_0\right)\beta_m I_f^{***} - \mu_a\left(\mu_a + \mu_f\right),$$

$$c = \mu_a\left(1 - \alpha_0\right) I_f^{***}, \tag{3.4}$$

then we can produce that $\triangle = b^2 - 4ac > 0$ if $\Re_{02} > 1$, thus, I_m^{***} has a unique positive root

$$I_m^{***} = \frac{-\left(1 - \alpha_0\right) I_f^{***}}{\beta_m S_m^{***} - \left(\mu_a + \mu_f\right)}. \tag{3.5}$$

The other equilibrium points are obtained from Eq.(2.5-2.8) as

$$S_h^{***} = \frac{\Lambda_h}{\beta_h \left(I_f^{***} + I_m^{***}\right) + \mu_h}, \; R_h^{***} = \frac{r I_h^{***}}{\psi \beta_h \left(I_f^{**} + I_m^{**}\right) + \mu_h}, \tag{3.6}$$

$$E_h^{***} = \frac{\beta_h \left(I_f^{***} + I_m^{***}\right) \left(S_h^{***} + \psi R_h^{***}\right)}{\left(\alpha_h + \mu_h\right)}, \; I_h^{***} = \frac{\alpha_h E_h^{***}}{\left(r + \mu_d + \mu_h\right)}. \tag{3.7}$$

Therefore, the endemic equilibrium

$$U^{***} = \left(S_f^{***},\ I_f^{***},\ S_m^{***},\ I_m^{***},\ S_h^{***},\ E_h^{***},\ I_h^{***},\ R_h^{***}\right)$$

is established whenever $\Re_{01} > 1$ and $\Re_{02} > 1$.

2. Now, we consider the condition $I_f = 0$, therefore, the model (2.1-2.8) reduces to

$$S_f' = \Lambda_f - \left(\mu_a + \alpha_0\right) S_f, \tag{3.8}$$

$$S_m' = \alpha_0 S_f - \beta_m S_m I_m - \mu_a S_m, \tag{3.9}$$

$$I_m' = \beta_m S_m I_m - \left(\mu_a + \mu_f\right) I_m, \tag{3.10}$$

$$S_h' = \Lambda_h - \beta_h I_m S_h - \mu_h S_h, \tag{3.11}$$

$$E_h' = \beta_h I_m \left(S_h + \psi R_h\right) - \left(\alpha_h + \mu_h\right) E_h, \tag{3.12}$$

$$I_h' = \alpha_h E_h - \left(r + \mu_d + \mu_h\right) I_h, \tag{3.13}$$

$$R_h' = r I_h - \psi \beta_h I_m R_h - \mu_h R_h. \tag{3.14}$$

We obtain from Eq.(3.8) and (3.9) that

$$S_f^* = \frac{\Lambda_f}{\left(\alpha_0 + \mu_a\right)}, \; S_m^* = \frac{\left(\mu_a + \mu_f\right)}{\beta_m}. \tag{3.15}$$

By using Eq.(3.15) in Eq.(3.10) then

$$I_m^* = \frac{\mu_a}{\beta_m} \left(\Re_{02} - 1\right), \tag{3.16}$$

from which it is essential for $\Re_{02} > 1$. We obtain the other equilibria from Eq.(3.11-3.14) as

$$S_h^* = \frac{\Lambda_h}{\beta_h I_m^* + \mu_h}, \; E_h^* = \frac{\beta_h I_m^* \left(S_h^* + \psi R_h^*\right)}{\left(\alpha_h + \mu_h\right)}, \; I_h^* = \frac{\alpha_h E_h^*}{\left(r + \mu_d + \mu_h\right)}, \; R_h^* = \frac{r I_h^*}{\psi \beta_h I_m^* + \mu_h}.$$

Therefore, the boundary equilibrium $U^* = \left(S_f^*, 0, S_m^*, I_m^*, S_h^*, E_h^*, I_h^*, R_h^*\right)$ is established whenever $\Re_{01} < 1$ and $\Re_{02} > 1$.

3. Again, we consider a case where all the means of transmission of the disease between the poultry in market is negligible including all infected poultry in farm considered killed such that only susceptible poultry goes to market i.e. $I_m = 0$ and $(1-\alpha_0) I_f = 0$. The model (2.1-2.8) will then reduce to

$$S_f' = \Lambda_f - \beta_f S_f I_f - (\mu_a + \alpha_0) S_f, \tag{3.17}$$
$$I_f' = \beta_f S_f I_f - (\mu_a + \mu_f) I_f, \tag{3.18}$$
$$S_m' = \alpha_0 S_f - \mu_a S_m, \tag{3.19}$$
$$S_h' = \Lambda_h - \beta_h I_f S_h - \mu_h S_h, \tag{3.20}$$
$$E_h' = \beta_h I_f (S_h + \psi R_h) - (\mu_h + \alpha_h) E_h, \tag{3.21}$$
$$I_h' = \alpha_h E_h - (r + \mu_d + \mu_h) I_h, \tag{3.22}$$
$$R_h' = r I_h - \psi \beta_h I_f R_h - \mu_h R_h. \tag{3.23}$$

We obtain from Eq.(3.18) and Eq.(3.17) that

$$S_f^{**} = \frac{(\mu_a + \mu_f)}{\beta_f} \tag{3.24}$$
$$I_f^{**} = \frac{(\alpha_0 + \mu_a)}{\beta_f}\left(\frac{1 - \alpha_0 + \mu_a + \mu_f}{\mu_a + \mu_f}\Re_{01} - 1\right). \tag{3.25}$$

On solving Eq.(3.19) and substituting S_f^{**},

$$S_m^{**} = \frac{\alpha_0 (\mu_a + \mu_f)}{\mu_a}. \tag{3.26}$$

From Eq.(3.20 - 3.23), we have

$$S_h^{**} = \frac{\Lambda_h}{\beta_h I_f^{**} + \mu_h}, E_h^{**} = \frac{\beta_h I_f^{**}(S_h^{**} + \psi R_h^{**})}{(\alpha_h + \mu_h)}, I_h^{**} = \frac{\alpha_h E_h^{**}}{(r + \mu_d + \mu_h)},$$

$$R_h^{**} = \frac{r I_h^{**}}{\psi \beta_h I_f^{**} + \mu_h}. \tag{3.27}$$

Therefore, the boundary equilibrium

$$U^{**} = \left(S_f^{**},\ I_f^{**},\ S_m^{**},\ 0,\ S_h^{**},\ E_h^{**},\ I_h^{**},\ R_h^{**}\right)$$

is established whenever $\Re_{01} > 1$ and $\Re_{02} < 1$. □

4 Stability of Equilibria

We have established the existence of disease-free and positive equilibria. We further investigate the stability of these equilibria.

4.1 Stability of the disease-free equilibrium

Theorem 2. *Let* $U^0 = \left(S_f^0, 0, S_m^0, 0, S_h^0, 0, 0, 0\right)$ *be the disease-free equilibrium of system (2.1-2.8). Then* U^0 *is locally asymptotically stable if* $\Re_0 < 1$, *but unstable if* $\Re_0 > 1$.

Proof. The characteristics equation of the Jacobian matrix at the disease-free equilibrium U^0 is

$$\begin{aligned}&(\lambda + \alpha_0 + \mu_a)(\lambda + \mu_a)(\lambda + \mu_h)(\lambda + \mu_h)(\lambda + \alpha_h + \mu_h)(\lambda + r + \mu_d + \mu_h)\\&\left(\lambda + 1 - \alpha_0 + \mu_a + \mu_f - \beta_f S_f^0\right)\left(\lambda + \mu_a + \mu_f - \beta_m S_m^0\right) = 0\end{aligned}$$

and the eigenvalues are $\lambda_1 = -(\alpha_0 + \mu_a)$, $\lambda_2 = -\mu_a$, $\lambda_3 = \lambda_4 = -\mu_h$, $\lambda_5 = -(\alpha_h + \mu_h)$, $\lambda_6 = -(r + \mu_d + \mu_h)$, $\lambda_7 = (1 - \alpha_0 + \mu_a + \mu_f)(\Re_{01} - 1)$, $\lambda_8 = (\mu_a + \mu_f)(\Re_{02} - 1)$. Clearly, all eigenvalues have negative real parts if $\Re_{01} < 1$ and $\Re_{02} < 1$ and consequently, $\Re_0 = max\{\Re_{01}, \Re_{02}\} < 1$. Thus, the disease-free equilibrium U^0 is locally asymptotically stable if $\Re_0 < 1$ but unstable if $\Re_0 > 1$. □

Theorem 3. *For system (2.1 - 2.8), if* $\Re_{01} < 1$, *the disease-free equilibrium* U^0 *is globally asymptotically stable.*

Proof. We shall construct this proof by taking the three subsystems one after the other

Poultry subsystem in farms: Define a Lyapunov function for the poultry subsystem in farms

$$L_{11} = S_f - S_f^0 - S_f^0 \ln\left(\frac{S_f}{S_f^0}\right) + I_f. \tag{4.1}$$

It follows that the derivative of L_{11} is

$$L'_{11} = -\frac{(\mu_a + \alpha_0)}{S_f}\left(S_f - S_f^0\right)^2 + (1-\alpha_0+\mu_a+\mu_f)\left(\Re_0 - 1\right) I_f \leq 0 \text{ if } \Re_{01} < 1.$$

Thus,

$$\begin{aligned}\Psi_1 &= \left\{(S_f,\ I_f) \in \mathbb{R}_+^2 : L'_{11} = 0\right\} \\ &= \left\{(S_f,\ I_f) \in \mathbb{R}_+^2 : S_f = S_f^0,\ I_f = 0\right\} = \left\{U_f^0\right\},\end{aligned}$$

which according to Lassale's invariace principle, U_f^0 is globally asymptotically stable [18, 19].

Poultry subsystem of markets: The poultry subsystem of markets with the avian components of farms already at the disease-free steady state is

$$S'_m = \alpha_0 S_f^0 - \beta_m S_m I_m - \mu_a S_m, \tag{4.2}$$

$$I'_m = \beta_m S_m I_m - (\mu_a + \mu_f) I_m, \tag{4.3}$$

we define a Lyapunov function as

$$L_{12} = S_m - S_m^0 - S_m^0 \ln\left(\frac{S_m}{S_m^0}\right) + I_m. \tag{4.4}$$

It follows that

$$L'_{12} = -\frac{\mu_a}{S_m}\left(S_m - S_m^0\right)^2 + (\mu_a + \mu_f)\left(\Re_{02} - 1\right) I_m \leq 0 \text{ if } \Re_{02} < 1.$$

Thus,

$$\begin{aligned}\Psi_2 &= \left\{(S_m,\ I_m) \in \mathbb{R}_+^2 : L'_{12} = 0\right\} \\ &= \left\{(S_m,\ I_m) \in \mathbb{R}_+^2 : S_m = S_m^0,\ I_m = 0\right\} = \left\{U_m^0\right\},\end{aligned}$$

which according to Lassale's invariace principle, U_m^0 is globally asymptotically stable [18, 19].

Human subsystem: Finally, we consider the human subsystem with the avian components already at the disease-free steady states.

$$\begin{aligned} S_h' &= \Lambda_h - \mu_h S_h, \\ E_h' &= -(\alpha_h + \mu_h)E_h, \\ I_h' &= \alpha_h E_h - (r + \mu_d + \mu_h)I_h, \\ R_h' &= rI_h - \mu_h R_h, \end{aligned} \tag{4.5}$$

we define a Lyapunov function

$$L_{13} = S_h - S_h^0 - S_h^0 \ln\left(\frac{S_h}{S_h^0}\right) + E_h + I_h + R_h, \tag{4.6}$$

then, it follows that the derivative of L_{13} along the solution of Eq.(4.6) is

$$L_{13}' = -\frac{\mu_h}{S_h}\left(S_h - S_h^0\right)^2 - \mu_h E_h - (\mu_d + \mu_h) I_h - \mu_h R_h.$$

Thus,

$$\begin{aligned} \Psi_3 &= \left\{(S_h,\ E_h,\ I_h,\ R_h) \in \mathbb{R}_+^4 : L_{13}' = 0\right\} \\ &= \left\{(S_h,\ E_h,\ I_h,\ R_h) \in \mathbb{R}_+^4 : S_h = S_h^0,\ E_h = 0,\ I_h = 0,\ R_h = 0\right\} = \left\{U_h^0\right\}, \end{aligned}$$

which according to Lassale's invariace principle, U_h^0 is globally asymptotically stable [18, 19].

□

4.2 Stability of the boundary equilibrium and the endemic equilibrium

The characteristics equation of the Jacobian matrix of 2.1 is obtained as

$$\begin{aligned} &\left[\beta_f^2 S_f I_f + (\lambda + \alpha_0 + \mu_a + \beta_f I_f)(\lambda + 1 - \alpha_0 + \mu_f + \mu_a - \beta_f S_f)\right] \\ &\times \left[\beta_m I_m (\lambda + \mu_a + \mu_f) + (\lambda + \mu_a)(\lambda - \beta_m S_m + \mu_a + \mu_f)\right](-\lambda - \beta_h (I_f + I_m) - \mu_h) \\ &\times \left[(r\psi\alpha_h\beta_h (I_f + I_m) - (\lambda + \alpha_h + \mu_h)(\lambda + \psi\beta_h (I_f + I_m) + \mu_h)(\lambda + r + \mu_d + \mu_h))\right] = 0. \end{aligned}$$

Theorem 4. *For system (2.1 - 2.8), the boundary equilibrium U^* is locally asymptotically stable whenever $\Re_{01} < 1$ and $\Re_{02} > 1$, the boundary equilibrium U^{**} is locally asymptotically stable whenever $\Re_{01} > 1$ and $\Re_{02} < 1$ and the endemic equilibrium U^{***} is locally asymptotically stable whenever $\Re_{01} > 1$ and $\Re_{02} > 1$.*

Proof. We have the proof as follows:

1. For the boundary equilibrium $U^* = \left(S_f^*, 0, S_m^*, I_m^*, S_h^*, E_h^*, I_h^*, R_h^*\right)$, one of the eigenvalues is

$$\lambda_1 = -\mu_a \frac{\beta_h}{\beta_m}\left(\Re_{01} - 1\right) - \mu_h \tag{4.7}$$

four of the eigenvalues are obtained from the two quadratic equations,

$$\lambda^2 + (1 - \alpha_0 + \mu_a + \mu_f)\left(\frac{(\alpha_0 + \mu_a)}{(1 - \alpha_0 + \mu_a + \mu_f)} + 1 - \Re_{01}\right)\lambda$$
$$+ (\alpha_0 + \mu_a)(1 - \alpha_0 + \mu_a + \mu_f)(1 - \Re_{01}) = 0 \tag{4.8}$$
$$\lambda^2 + \mu_a \Re_{02}\lambda + \mu_a(\mu_a + \mu_f)(\Re_{02} - 1) = 0 \tag{4.9}$$

and the other three eigenvalues are obtained from the cubic equation

$$\begin{aligned}&\lambda^3 + (\alpha_h + r + \mu_d + 3\mu_h + \psi\beta_h I_m^*)\lambda^2\\&+ ((\psi\beta_h I_m^* + \mu_h)(\alpha_h + 2\mu_h + r + \mu_d))\lambda\\&+ (\psi\beta_h I_m^* + \mu_h)(\alpha_h + \mu_h)(r + \mu_d + \mu_h) - \alpha_h r\psi\beta_h I_m^* = 0,\end{aligned}$$

from which, if $\Re_{01} < 1,\ \Re_{02} > 1$, all the eigenvalues have negative real parts.

2. For the boundary equilibrium $U^{**} = \left(S_f^{**}, I_f^{**}, S_m^{**}, I_m^{**}, S_h^{**}, E_h^{**}, I_h^{**}, R_h^{**}\right)$, one of the eigenvalues is

$$\lambda_1 = -\frac{\beta_h}{\beta_f}(\alpha_0 + \mu_a)\left(\frac{1 - \alpha_0 + \mu_a + \mu_f}{\mu_a + \mu_f}\Re_{01} - 1\right) - \mu_h \tag{4.10}$$

four of the eigenvalues can be obtained from the following two quadratic equations

$$\lambda^2 + \left(\frac{(\alpha_0+\mu_a)(1-\alpha_0+\mu_a+\mu_f)}{\mu_a+\mu_f}\Re_{01} + (1-\alpha_0)\right)\lambda$$
$$+ (\alpha_0+\mu_a)\left(\frac{1-\alpha_0+\mu_a+\mu_f}{\mu_a+\mu_f}\Re_{01} - 1\right) = 0, \tag{4.11}$$

$$\lambda^2 + \left(\mu_a^2 + (\mu_a+\mu_f)(\mu_a - \alpha_0\beta_m)\right)\lambda + (\mu_a+\mu_f)(\mu_a-\alpha_0\beta_m) = 0. \tag{4.12}$$

The remaining three eigenvalues are obtained from the cubic equations

$$(\alpha_h+\mu_h)(r+\mu_d+\mu_h)\psi\beta_h I_f^{**} + \mu_h(\alpha_h+\mu_h)(r+\mu_d+\mu_h) + \alpha_h r\psi\beta_h I_f^{**}$$
$$+ \left(\psi\beta_h I_f^{**}(\alpha_h+2\mu_h+r+\mu_d) + \mu_h(\alpha_h+2\mu_h+r+\mu_d) + (\alpha_h+\mu_h)(r+\mu_d+\mu_h)\right)\lambda$$
$$+ \left(\psi\beta_h I_f^{**} + \mu_h + (\alpha_h+2\mu_h+r+\mu_d)\right)\lambda^2 + \lambda^3 = 0$$

from which, if $\Re_{01} > 1$, $\Re_{02} < 1$, all the eigenvalues have negative real parts.

3. For the boundary equilibrium

$$U^{***} = \left(S_f^{***},\ I_f^{***}\ S_m^{***},\ I_m^{***},\ S_h^{***},\ E_h^{***},\ I_h^{***},\ R_h^{***}\right),$$

one of the eigenvalues is

$$\lambda_1 = -\beta_h I_f^{***} - \beta_h I_m^{***} - \mu_h \tag{4.13}$$

four of the eigenvalues can be obtained from the two quadratic equations

$$\lambda^2 + \left(\beta_f I_f^{***} - \beta_f S_f^{***} + (1-\alpha_0+\mu_a+\mu_f) + (\alpha_0+\mu_a)\right)\lambda$$
$$+ (\alpha_0+\mu_a)(1-\alpha_0+\mu_a+\mu_f) - (\alpha_0+\mu_a)\beta_f S_f^{***}$$
$$+ \beta_f I_f^{***}(1-\alpha_0+\mu_a+\mu_f) = 0,$$
$$\lambda^2 + (\beta_m I_m^{***} - \beta_m S_m^{***} + \mu_a + (\mu_a+\mu_f))\lambda$$
$$+ (\mu_a+\mu_f)\beta_m I_m^{***} - \mu_a\beta_m S_m^{***} + \mu_a(\mu_a+\mu_f) = 0. \tag{4.14}$$

The remaining three eigenvalues are obtained from the cubic equations

$$\lambda^3 + \left(\alpha_h+\mu_d+r+3\mu_h+\psi\beta_h I_f^{***} + \psi\beta_h I_m^{***}\right)\lambda^2$$
$$+ \left((\alpha_h+2\mu_h+r+\mu_d)\left(\psi\beta_h I_f^{***} + \psi\beta_h I_m^{***} + \mu_h\right) + (\alpha_h+\mu_h)(r+\mu_d+\mu_h)\right)\lambda$$
$$+ \left(\psi\beta_h I_f^{***} + \psi\beta_h I_m^{***} + \mu_h\right)(\alpha_h+\mu_h)(r+\mu_d+\mu_h) - \alpha_h r\left(\psi\beta_h I_f^{***} + \psi\beta_h I_m^{***}\right)$$
$$= 0 \tag{4.15}$$

from which, if $\Re_{01} > 1,\ \Re_{02} > 1$, all the eigenvalues have negative real parts.

□

Theorem 5. *For system (2.1 - 2.8),*

1. *the boundary equilibrium* $U^* = \left(S_f^*,\ 0,\ S_m^*,\ I_m^*,\ S_h^*,\ E_h^*,\ I_h^*,\ R_h^*\right)$ *is globally asymptotically stable whenever* $\Re_{01} < 1,\ \Re_{02} > 1$.
2. *the boundary equilibrium* $U^{**} = \left(S_f^{**},\ I_f^{**},\ S_m^{**},\ 0,\ S_h^{**},\ E_h^{**},\ I_h^{**},\ R_h^{**}\right)$ *is globally asymptotically stable whenever* $\Re_{01} > 1,\ \Re_{02} < 1$.
3. *the endemic equilibrium* $U^{***} = \left(S_f^{***},\ I_f^{***},\ S_m^{***},\ I_f^{***},\ S_h^{***},\ E_h^{***},\ I_h^{***},\ R_h^{***}\right)$ *is globally asymptotically stable whenever* $\Re_{01} > 1,\ \Re_{02} > 1$.

Proof. We consider the global stability of the boundary equilibrium and the endemic equilibrium.

1. The boundary equilibrium U^*.

   (a) Consider the poultry subsystem in farms and define a Lyapunov function
   $$L_{21} = S_f - S_f^* - S_f^* \ln\left(\frac{S_f}{S_f^*}\right) + I_f \tag{4.16}$$
   then the derivative of L_{21} along the solution of Eq.(2.1) and Eq.(2.2) is
   $$L_{21}' = -\frac{(\mu_a + \alpha_0)}{S_f}\left(S_f - S_f^*\right)^2 + (1 - \alpha_0 + \mu_a + \mu_f)\,(\Re_0 - 1)\, I_f$$

$$\begin{aligned}\Psi_4 &= \left\{(S_f,\ I_f) \in \mathbb{R}_+^2 : L_{21}' = 0\right\} = \left\{(S_f,\ I_f) \in \mathbb{R}_+^2 : S_f = S_f^*,\ I_f = 0\right\}\\ &= \left\{U_f^*\right\},\end{aligned}$$

which according to Lassale's invariace principle, U_f^* is globally asymptotically stable [18, 19].

(b) Next, we consider the poultry subsystem of markets with the avian components of farms already at the disease-free steady state

$$S_m' = \alpha_0 S_f^* - \beta_m S_m I_m - \mu_a S_m \tag{4.17}$$

$$I_m' = \beta_m S_m I_m - (\mu_a + \mu_f) I_m \tag{4.18}$$

we define a Lyapunov function

$$L_{22} = S_m - S_m^* - S_m^* \ln\left(\frac{S_m}{S_m^*}\right) + I_m - I_m^* - I_m^* \ln\left(\frac{I_m}{I_m^*}\right) \tag{4.19}$$

and then the derivative of L_{22} along the soluion of Eq.(4.17) and Eq.(4.18) is

$$L_{22}' = \mu_a \Re_{02} S_m^* \left(2 - \frac{S_m}{S_m^*} - \frac{S_m^*}{S_m}\right).$$

Since $2 - \frac{S_m}{S_m^*} - \frac{S_m^*}{S_m} \leq 0$ if $\Re_{02} > 1$, then $L_{22}' \leq 0$, and thus

$$\begin{aligned}\Psi_5 &= \left\{(S_m,\ I_m) \in \mathbb{R}_+^2 : L_{22}' = 0\right\} = \left\{(S_m,\ I_m) \in \mathbb{R}_+^2 : S_m = S_m^*,\ I_m = I_m^*\right\} \\ &= \{U_m^*\},\end{aligned}$$

which according to Lassale's invariace principle, U_m^* is globally asymptotically stable [18, 19].

(c) Finally, we are considering the human subsystem with the avian components of markets already at the endemic steady state.

$$\begin{aligned} S_h' &= \Lambda_h - \beta_h S_h I_m^* - \mu_h S_h \\ E_h' &= \beta_h S_h I_m^* + \psi \beta_h R_h I_m^* - (\alpha_h + \mu_h) E_h \\ I_h' &= \alpha_h E_h - (r + \mu_h + \mu_d) I_h \\ R_h' &= r I_h - \psi \beta_h R_h I_m^* - \mu_h R_h \end{aligned} \tag{4.20}$$

we define the Lyapunov function as

$$\begin{aligned} L_{23} =& S_h - S_h^* - S_h^* \ln\left(\frac{S_h}{S_h^*}\right) + E_h - E_h^* - E_h^* \ln\left(\frac{E_h}{E_h^*}\right) \\ &+ I_h - I_h^* - I_h^* \ln\left(\frac{I_h}{I_h^*}\right) + R_h - R_h^* - R_h^* \ln\left(\frac{R_h}{R_h^*}\right) \end{aligned} \tag{4.21}$$

and then the derivative of L_{23} along the solutions of (4.20) is

$$\begin{aligned} L'_{23} =& \mu_a\left(\Re_{02}-1\right)\left(3-\frac{S_h^*}{S_h}-\frac{E_h}{E_h}-\frac{S_h}{S_h^*}\frac{E_h^*}{E_h}\right) \\ &+\mu_a\psi R_h^*\left(\Re_{02}-1\right)\left(2-\frac{E_h}{E_h^*}+\frac{I_h}{I_h^*}-\frac{E_h^*}{E_h}\frac{R_h}{R_h^*}-\frac{I_h}{I_h^*}\frac{R_h^*}{R_h}\right) \\ &+\mu_h S_h^*\left(2-\frac{S_h^*}{S_h}-\frac{S_h}{S_h^*}\right)+\alpha_h E_h^*\left(1+\frac{E_h}{E_h^*}-\frac{I_h}{I_h^*}-\frac{I_h^*}{I_h}\right) \\ &+\mu_h R_h^*\left(1+\frac{I_h}{I_h^*}-\frac{R_h}{R_h^*}-\frac{I_h}{I_h^*}\frac{R_h^*}{R_h}\right). \end{aligned} \tag{4.22}$$

Since $3-\frac{S_h^*}{S_h}-\frac{E_h}{E_h}-\frac{S_h}{S_h^*}\frac{E_h^*}{E_h}\leq 0$, $2-\frac{E_h}{E_h^*}+\frac{I_h}{I_h^*}-\frac{E_h^*}{E_h}\frac{R_h}{R_h^*}-\frac{I_h}{I_h^*}\frac{R_h^*}{R_h}\leq 0$, $2-\frac{S_h^*}{S_h}-\frac{S_h}{S_h^*}\leq 0$, $1+\frac{E_h}{E_h^*}-\frac{I_h}{I_h^*}-\frac{I_h^*}{I_h}\leq 0$, $1+\frac{I_h}{I_h^*}-\frac{R_h}{R_h^*}-\frac{I_h}{I_h^*}\frac{R_h^*}{R_h}\leq 0$, if $\Re_{02}>1$, then $L'_{23}\leq 0$, and thus,

$$\begin{aligned} \Psi_6 &= \left\{(S_h,\ E_h,\ I_h,\ R_h)\in\mathbb{R}_+^4 : L'_{23}=0\right\} \\ &= \left\{(S_h,\ E_h,\ I_h,\ R_h)\in\mathbb{R}_+^4 : S_h=S_h^*,\ E_h=E_h^*,\ I_h=I_h^*,\ R_h=R_h^*\right\} \\ &= \left\{U_h^*\right\}, \end{aligned}$$

which according to Lassale's invariace principle, U_h^* is globally asymptotically stable [18, 19]. In conclusion, if $\Re_{01}<1$, $\Re_{02}>1$, the boundary equilibrium $U^*$is globally asymptotically stable.

2. The boundary equilibrium U^{**}

   (a) We firstly consider the poultry subsystem in farm and define a Lyapunov function

$$L_{31}=S_f-S_f^{**}-S_f^{**}\ln\left(\frac{S_f}{S_f^{**}}\right)+I_f-I_f^{**}-I_f^{**}\ln\left(\frac{I_f}{I_f^{**}}\right), \tag{4.23}$$

then the derivative of L_{31} along solutions of system (3.17) is obtain as

$$L'_{31}=-\frac{(\alpha_0+\mu_a)}{S_f}\left(S_f-S_f^{**}\right)^2+\beta_f S_f^{**}I_f^{**}\left(2-\frac{S_f^{**}}{S_f}-\frac{S_f}{S_f^{**}}\right), \tag{4.24}$$

since $2 - \frac{S_f^{**}}{S_f} - \frac{S_f}{S_f^{**}} \leq 0$, $L_{31}' \leq 0$. Thus,

$$\begin{aligned}\Psi_7 &= \left\{(S_f, I_f) \in \mathbb{R}_+^2 : L_{31}' = 0\right\} \\ &= \left\{(S_f,\ I_f) \in \mathbb{R}_+^2 : S_f = S_f^{**},\ I_f = I_f^{**}\right\} = \left\{U_f^{**}\right\},\end{aligned}$$

which according to Lassale's invariace principle, U_f^{**} is globally asymptotically stable [18, 19].

(b) Next, we consider the poultry subsystem of markets with avian components of farm at the disease-free steady state

$$S_m' = \alpha_0 S_f^{**} - \beta_m S_m I_m - \mu_a S_m, \tag{4.25}$$

$$I_m' = \beta_m S_m I_m - (\mu_a + \mu_f)\, I_m. \tag{4.26}$$

We define a Lyapunov function

$$L_{32} = S_m - S_m^{**} - S_m^{**} ln \frac{S_m}{S_m^{**}} + I_m, \tag{4.27}$$

then the derivative of L_{32} along the solution of system (4.25) and (4.26) is

$$L_{32}' = \mu_a S_m^{**}\left(2 - \frac{S_m^{**}}{S_m} - \frac{S_m}{S_m^{**}}\right), \tag{4.28}$$

since $2 - \frac{S_m^{**}}{S_m} - \frac{S_m}{S_m^{**}} \leq 0$, $L_{32}' \leq 0$, thus,

$$\begin{aligned}\Psi_8 &= \left\{(S_m,\ I_m) \in \mathbb{R}_+^2 : L_{32}' = 0\right\} \\ &= \left\{(S_m,\ I_m) \in \mathbb{R}_+^2 : S_m = S_m^{**},\ I_m = I_m^{**}\right\} = \left\{U_m^{**}\right\},\end{aligned}$$

which according to Lassale's invariace principle, U_m^{**} is globally asymptotically stable [18, 19].

(c) Lastly, considering the human subsystem with the avian components already at the endemic steady state

$$\begin{aligned}S_h' &= \Lambda_h - \beta_h S_h I_f^{**} - \mu_h S_h, \\ E_h' &= \beta_h I_f^{**}\left(S_h + \psi R_h\right) - (\alpha_h + \mu_h)\, E_h, \\ I_h' &= \alpha_h E_h - (r + \mu_d + \mu_h)\, I_h, \\ R_h' &= r I_h - \psi \beta_h I_f^{**} R_h - \mu_h R_h.\end{aligned} \tag{4.29}$$

We define a Lyapunov function

$$L_{33} = S_h - S_h^{**} - S_h^{**}\ln\left(\frac{S_h}{S_h^{**}}\right) + E_h - E_h^{**} - E_h^{**}\ln\left(\frac{E_h}{E_h^{**}}\right) + I_h - I_h^{**} - I_h^{**}\ln\left(\frac{I_h}{I_h^{**}}\right) + R_h - R_h^{**} - R_h^{**}\ln\left(\frac{R_h}{R_h^{**}}\right) \quad (4.30)$$

and then, the derivative of L_{33} along the solutions of system (4.30) is

$$\begin{aligned} L'_{33} =& \beta_h I_f^{**} S_h^{**}\left(3 - \frac{S_h^{**}}{S_h} - \frac{E_h}{E_h^{**}} - \frac{S_h}{S_h^{**}}\frac{E_h^{**}}{E_h}\right) \\ & + \mu_h R_h\left(1 + \frac{I_h}{I_h^{**}} - \frac{R_h}{R_h^{**}} - \frac{I_h}{I_h^{**}}\frac{R_h^{**}}{R_h}\right) \\ & + \psi\beta_h I_f^{**} R_h^{**}\left(2 - \frac{E_h}{E_h^{**}} - \frac{E_h^{**}}{E_h}\frac{R_h}{R_h^{**}} + \frac{I_h}{I_h^{**}} - \frac{I_h}{I_h^{**}}\frac{R_h^{**}}{R_h}\right) \\ & + \mu_h S_h^{**}\left(2 - \frac{S_h}{S_h^{**}} - \frac{S_h^{**}}{S_h}\right) \\ & + \alpha_h E_h^{**}\left(1 + \frac{E_h}{E_h^{**}} - \frac{I_h}{I_h^{**}} - \frac{E_h}{E_h^{**}}\frac{I_h^{**}}{I_h}\right). \end{aligned} \quad (4.31)$$

Since $3 - \frac{S_h^{**}}{S_h} - \frac{E_h}{E_h^{**}} - \frac{S_h}{S_h^{**}}\frac{E_h^{**}}{E_h} \leq 0$, $2 - \frac{E_h}{E_h^{**}} - \frac{E_h^{**}}{E_h}\frac{R_h}{R_h^{**}} + \frac{I_h}{I_h^{**}} - \frac{I_h}{I_h^{**}}\frac{R_h^{**}}{R_h} \leq 0$, $2 - \frac{S_h}{S_h^{**}} - \frac{S_h^{**}}{S_h} \leq 0$, $1 + \frac{E_h}{E_h^{**}} - \frac{I_h}{I_h^{**}} - \frac{E_h}{E_h^{**}}\frac{I_h^{**}}{I_h} \leq 0$, $1 + \frac{I_h}{I_h^{**}} - \frac{R_h}{R_h^{**}} - \frac{I_h}{I_h^{**}}\frac{R_h^{**}}{R_h} \leq 0$, $L'_{33} \leq 0$, and thus,

$$\begin{aligned} \Psi_9 &= \left\{(S_h, E_h, I_h, R_h) \in \mathbb{R}_+^4 : L'_{33} = 0\right\} \\ &= \left\{(S_h, E_h, I_h, R_h) \in \mathbb{R}_+^4 : S_h = S_h^{**}, E_h = E_h^{**}, I_h = I_h^{**}, R_h = R_h^{**}\right\} \\ &= \left\{U_h^{**}\right\}, \end{aligned}$$

which according to Lassale's invariace principle, U_h^{**} is globally asymptotically stable [18, 19]. In conclusion, if $\Re_{01} < 1$, $\Re_{02} > 1$, the boundary equilibrium U^{**} is globally asymptotically stable.

3. **The endemic equilibrium U^{***}**

   (a) We first consider the poultry subsystem in farms and define a Lyapunov function

$$L_{41} = S_f - S_f^{***} - S_f^{***}\ln\left(\frac{S_f}{S_f^{***}}\right) + I_f - I_f^{***} - I_f^{***}\ln\left(\frac{I_f}{I_f^{***}}\right) \quad (4.32)$$

and then the derivative of L_{31} is

$$L'_{41} = -\frac{(\mu_a + \alpha_0)}{S_f}\left(S_f - S_f^{***}\right)^2 + (\mu_a + \alpha_0)\left(\Re_{01} - 1\right) S_f^{***}\left(2 - \frac{S_f^{**}}{S_f} - \frac{S_f}{S_f^{**}}\right). \tag{4.33}$$

If $\Re_{01} > 1$, then $2 - \frac{S_f^{***}}{S_f} - \frac{S_f}{S_f^{***}} \leq 0$ and $L'_{41} \leq 0$ and thus,

$$\begin{aligned}\Psi_{10} &= \left\{(S_f, I_f) \in \mathbb{R}_+^2 : L'_{41} = 0\right\}\\ &= \left\{(S_f, I_f) \in \mathbb{R}_+^2 : S_f = S_f^{***}, I_f = I_f^{***}\right\}\\ &= \left\{U_f^{***}\right\}.\end{aligned}$$

According to Lassalle's invariance principle, U_f^{***} is globally asymptotically stable.

(b) Consider the poultry subsystem in market

$$S'_m = \alpha_0 S_f^{***} - \beta_m S_m I_m - \mu_a S_m \tag{4.34}$$

$$I'_m = \beta_m S_m I_m + (1 - \alpha_0) I_f^{***} - (\mu_a + \mu_f) I_m \tag{4.35}$$

and define a Lyapunov function

$$L_{42} = S_m - S_m^{***} - S_m^{***}\ln\left(\frac{S_m}{S_m^{***}}\right) + I_m - I_m^{***} - I_m^{***}\ln\left(\frac{I_m}{I_m^{***}}\right) \tag{4.36}$$

then take the derivative of L_{42}

$$\begin{aligned}L'_{42} = {}& \frac{-\mu_a}{S_m}\left(S_m - S_m^{***}\right)^2 + \beta_m S_m^{***} I_m^{***}\left(2 - \frac{S_m^{***}}{S_m} - \frac{S_m}{S_m^{***}}\right)\\ &+ \frac{(1-\alpha_0)(\mu_a + \alpha_0)}{\beta_f}\left(\Re_{01} - 1\right)\left(2 - \frac{I_m}{I_m^{***}} - \frac{I_m^{***}}{I_m}\right).\end{aligned}$$

If $\Re_{01} > 1$, $\Re_{02} > 1$, then $2 - \frac{S_m^{***}}{S_m} - \frac{S_m}{S_m^{***}} \leq 0$, $2 - \frac{I_m}{I_m^{***}} - \frac{I_m^{***}}{I_m} \leq 0$ and $L'_{42} \leq 0$ and thus

$$\begin{aligned}\Psi_{11} &= \left\{(S_m, I_m) \in \mathbb{R}_+^2 : L'_{42} = 0\right\}\\ &= \left\{(S_m, I_m) \in \mathbb{R}_+^2 : S_m = S_m^{***},\ I_m = I_m^{***}\right\}\\ &= \left\{U_m^{***}\right\}.\end{aligned}$$

According to Lassalle's invariance principle, U_m^{***} is globally asymptotically stable.

(c) Finally, consider the human subsystem

$$S_h' = \Lambda_h - \beta_h S_h I_f^{***} - \beta_h S_h I_m^{***} - \mu_h S_h \tag{4.37}$$

$$E_h' = \beta_h (I_f^{***} + I_m^{***})(S_h + \psi R_h) - (\alpha_h + \mu_h) E_h \tag{4.38}$$

$$I_h' = \alpha_h E_h - (r + \mu_d + \mu_h) I_h \tag{4.39}$$

$$R_h' = r I_h - \beta_h (I_f^{***} + I_m^{***}) \psi R_h - \mu_h R_h \tag{4.40}$$

and define a Lyapunov function

$$\begin{aligned} L_{43} = {} & S_h - S_h^{***} - S_h^{***} \ln\left(\frac{S_h}{S_h^{***}}\right) + E_h - E_h^{***} - E_h^{***} \ln\left(\frac{E_h}{E_h^{***}}\right) \\ & + I_h - I_h^{***} - I_h^{***} \ln\left(\frac{I_h}{I_h^{***}}\right) + R_h - R_h^{***} - R_h^{***} \ln\left(\frac{R_h}{R_h^{***}}\right) \end{aligned} \tag{4.41}$$

then take the derivative of L_{33} along the solution of the system (4.37 - 4.40)

$$\begin{aligned} L_{43}' = & -\frac{\mu_h}{S_h}\left(S_h - S_h^{***}\right)^2 + \beta_h S_h^{***}\left(I_f^{***} + I_m^{***}\right) \\ & \times \left(3 - \frac{S_h^{***}}{S_h} - \frac{E_h}{E_h^{***}} - \frac{S_h}{S_h^{***}}\frac{E_h^{***}}{E_h}\right) \\ & + \psi \beta_h R_h^{***}\left(I_f^{***} + I_m^{***}\right)\left(2 - \frac{E_h}{E_h^{***}} + \frac{I_h}{I_h^{***}} - \frac{E_h^{***}}{E_h}\frac{R_h}{R_h^{***}} - \frac{I_h}{I_h^{***}}\frac{R_h^{***}}{R_h}\right) \\ & + \mu_h R_h^{***}\left(1 - \frac{I_h}{I_h^{***}} - \frac{R_h}{R_h^{***}} + \frac{I_h}{I_h^{***}}\frac{R_h^{***}}{R_h}\right) \\ & + \alpha_h E_h^{***}\left(1 + \frac{E_h}{E_h^{***}} - \frac{I_h}{I_h^{***}} - \frac{E_h}{E_h^{***}}\frac{I_h^{***}}{I_h}\right). \end{aligned}$$

Since

$$\begin{aligned} \left(3 - \frac{S_h^{***}}{S_h} - \frac{E_h}{E_h^{***}} - \frac{S_h}{S_h^{***}}\frac{E_h^{***}}{E_h}\right) &\le 0, \\ \left(2 - \frac{E_h}{E_h^{***}} + \frac{I_h}{I_h^{***}} - \frac{E_h^{***}}{E_h}\frac{R_h}{R_h^{***}} - \frac{I_h}{I_h^{***}}\frac{R_h^{***}}{R_h}\right) &\le 0, \\ \left(1 - \frac{I_h}{I_h^{***}} - \frac{R_h}{R_h^{***}} + \frac{I_h}{I_h^{***}}\frac{R_h^{***}}{R_h}\right) &\le 0, \\ \left(1 + \frac{E_h}{E_h^{***}} - \frac{I_h}{I_h^{***}} - \frac{E_h}{E_h^{***}}\frac{I_h^{***}}{I_h}\right) &\le 0, \end{aligned}$$

if $\Re_{02} > 1$, then $L'_{43} \leq 0$ and thus,

$$\begin{aligned}\Psi_{12} &= \left\{(S_h, E_h, I_h, R_h) \in \mathbb{R}^4_+ : L'_{43} = 0\right\} \\ &= \left\{(S_h, E_h, I_h, R_h)) \in \mathbb{R}^4_+ : S_h = S_h^{***}, E_h = E_h^{***}, I_h = I_h^{***}, R_h = R_h^{***}\right\} \\ &= \{U_h^{***}\}.\end{aligned}$$

According to Lassalle's invariance principle, U_h^{***} is globally asymptotically stable.

□

5 Numerical Simulations

In this section, we present numerical simulations of model (2.1 - 2.8) by considering the parameters in the following examples to obtain the stability of the disease free-equilibrium, the boundary equilibria and the endemic equilibrium represented as a time-series diagram.

Example 1. Consider the parameters $\Lambda_f = 80$, $\beta_f = 0.0027$, $\mu_a = 0.17$, $\mu_f = 0.75$, $\alpha_0 = 0.86$, $\beta_m = 0.0018$, $\Lambda_h = 280$, $\beta_h = 0.0018$, $\mu_h = 0.69$, $\mu_d = 0.83$, $r = 0.61$, $\psi = 0.01$, $\alpha_h = 0.5$. Figure (5.1 - 5.2) gives the time-variation diagram of system (2.1 - 2.8). It is discovered that the disease-free equilibrium U^0 is globally asymptotically stable whenever $\Re_0 < 1$.

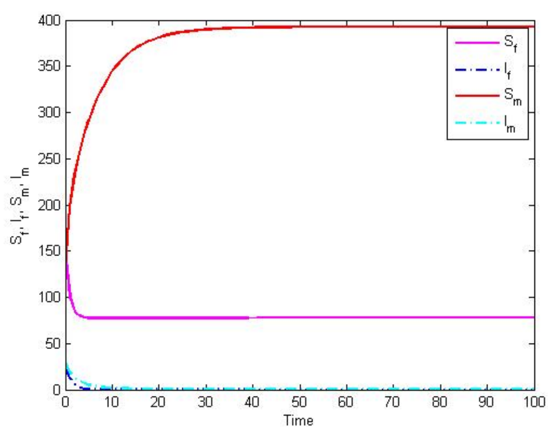

Figure 5.1: Time variation diagram of system (2.1) when $\Re_{01} < 1$ and $\Re_{02} < 1$.

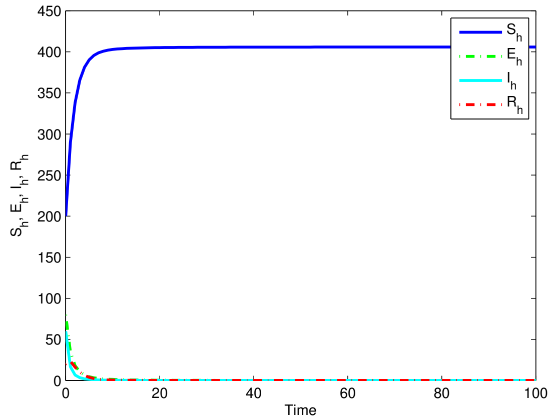

Figure 5.2: Time variation diagram of system (2.1) when $\Re_{01} < 1$ and $\Re_{02} < 1$.

Example 2. Consider the parameters $\Lambda_f = 80$, $\beta_f = 0.0027$, $\mu_a = 0.17$, $\mu_f = 0.75$, $\alpha_0 = 0.86$, $\beta_m = 0.0045$, $\Lambda_h = 280$, $\beta_h = 0.0018$, $\mu_h = 0.69$, $\mu_d = 0.83$, $r = 0.61$, $\psi = 0.01$, $\alpha_h = 0.5$. Figure (5.3 - 5.4) gives the time-variation diagram of system (2.1 - 2.8). It is discovered that the boundary equilibrium U^* is globally asymptotically stable whenever $\Re_{01} < 1$ and $\Re_{02} > 1$.

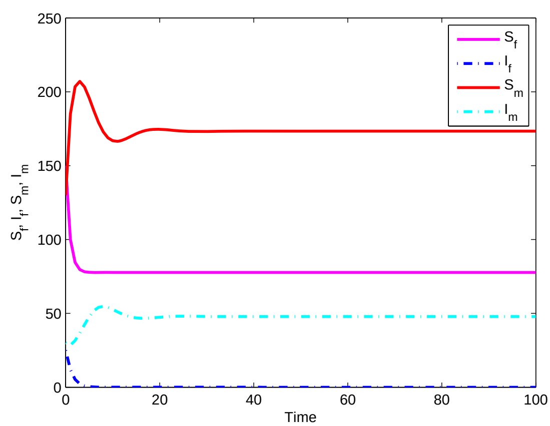

Figure 5.3: Time variation diagram of system (2.1) when $\Re_{01} < 1$ and $\Re_{02} > 1$.

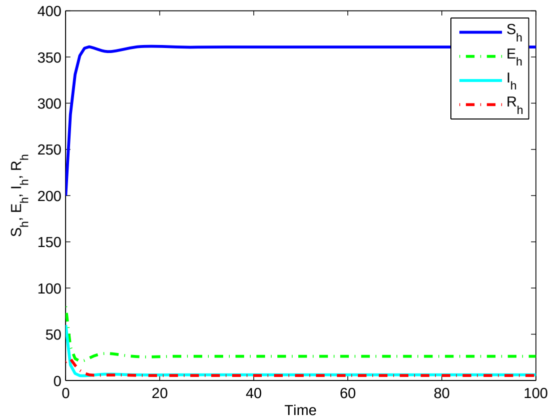

Figure 5.4: Time variation diagram of system (2.1) when $\Re_{01} < 1$ and $\Re_{02} > 1$.

Example 3. Consider the parameters $\Lambda_f = 80$, $\beta_f = 0.015$, $\mu_a = 0.17$, $\mu_f = 0.75$, $\alpha_0 = 0.86$, $\beta_m = 0.0018$, $\Lambda_h = 280$, $\beta_h = 0.0018$, $\mu_h = 0.69$, $\mu_d = 0.83$, $r = 0.61$, $\psi = 0.01$, $\alpha_h = 0.5$. Figure (5.5 - 5.6) gives the time-variation diagram of system (2.1 - 2.8). It is discovered that the boundary equilibrium U^{**} is globally asymptotically stable whenever $\Re_{01} > 1$ and $\Re_{02} < 1$.

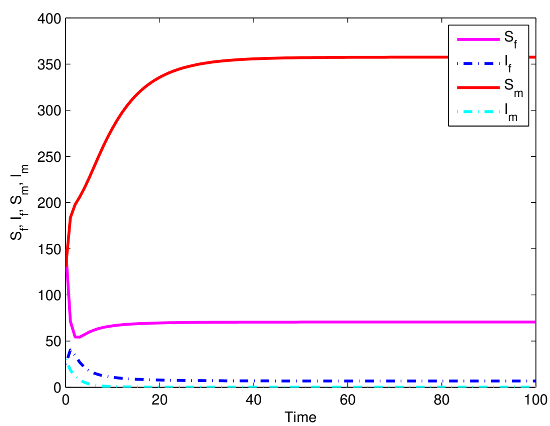

Figure 5.5: Time variation diagram of system (2.1) when $\Re_{01} > 1$ and $\Re_{02} < 1$.

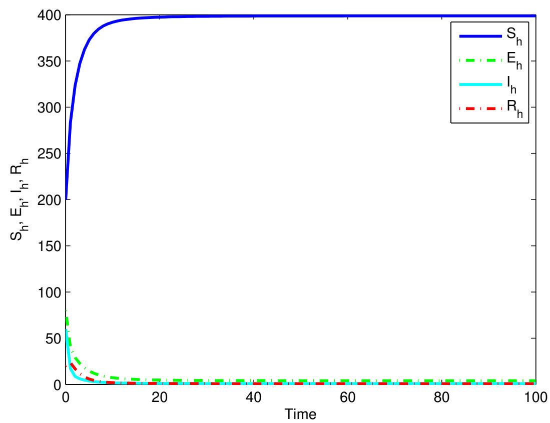

Figure 5.6: Time variation diagram of system (2.1) when $\Re_{01} > 1$ and $\Re_{02} < 1$.

Example 4. Consider the parameters $\Lambda_f = 80$, $\beta_f = 0.015$, $\mu_a = 0.17$, $\mu_f = 0.75$, $\alpha_0 = 0.86$, $\beta_m = 0.0045$, $\Lambda_h = 280$, $\beta_h = 0.0018$, $\mu_h = 0.69$, $\mu_d = 0.83$, $r = 0.61$, $\psi = 0.01$, $\alpha_h = 0.5$. Figure (5.7 - 5.8) gives the time-variation diagram of system (2.1 - 2.8). It is discovered that the endemic equilibrium U^{***} is globally asymptotically stable whenever $\Re_{01} > 1$ and $\Re_{02} > 1$.

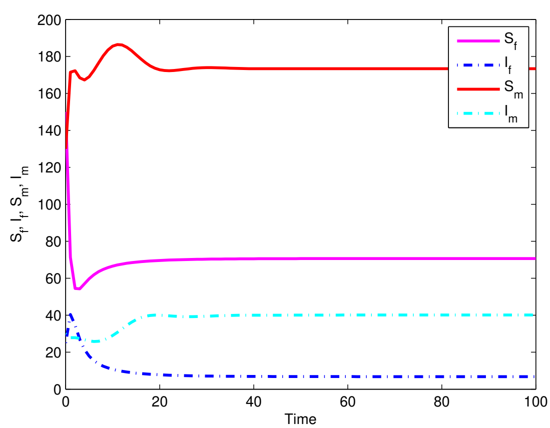

Figure 5.7: Time variation diagram of system (2.1) when $\Re_{01} > 1$ and $\Re_{02} > 1$.

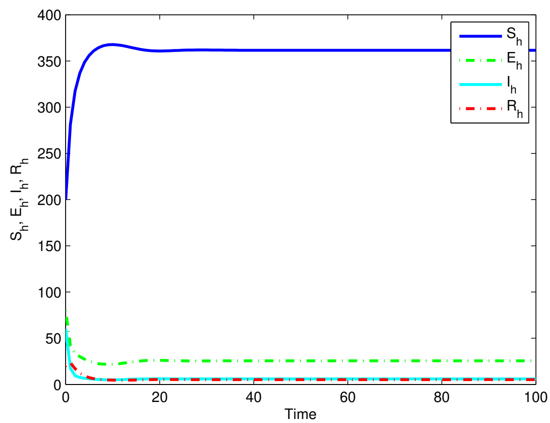

Figure 5.8: Time variation diagram of system (2.1) when $\Re_{01} > 1$ and $\Re_{02} > 1$.

Example 5. Consider the parameters $\Lambda_f = 80$, $\beta_f = 0.015$, $\mu_a = 0.17$, $\mu_f = 0.75$, $\alpha_0 = 0.86$, $\beta_m = 0.0045$, $\Lambda_h = 280$, $\mu_h = 0.69$, $\mu_d = 0.83$, $r = 0.61$, $\psi = 0.01$, $\alpha_h = 0.5$. Figure 5.9 shows the time-variation diagram of system E_h with time, with E_h increasing as β_h is increasing.

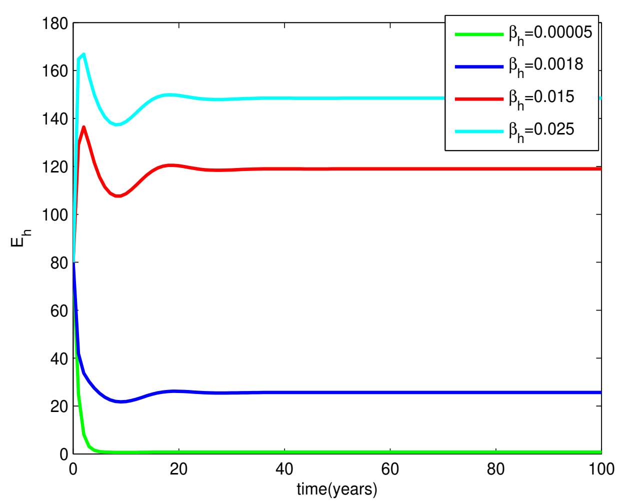

Figure 5.9: Time variation of E_h at various values of β_h.

Example 6. Consider the parameters $\Lambda_f = 80$, $\beta_f = 0.015$, $\mu_a = 0.17$, $\mu_f = 0.75$, $\alpha_0 = 0.86$, $\beta_m = 0.0045$, $\Lambda_h = 280$, $\beta_h = 0.0018$, $\mu_h = 0.69$, $\mu_d = 0.83$, $r = 0.61$, $\psi = 0.01$, $\alpha_h = 0.5$. Figure 5.10 gives the time-variation diagram of system I_m. It is observe that I_m decreases as α_0 increases.

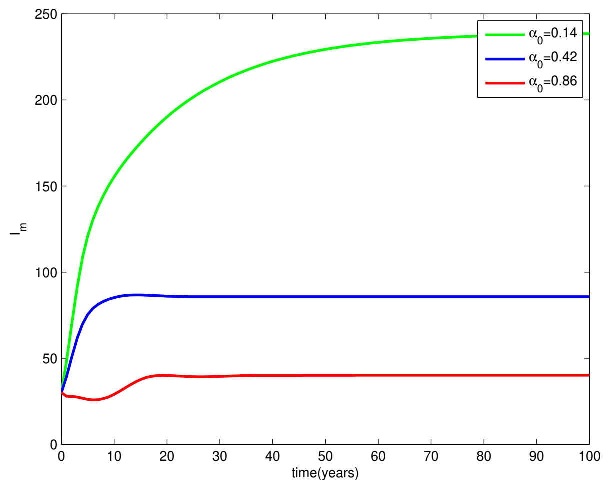

Figure 5.10: Time variation of I_m at various values of α_0.

However, in order to reduce the spread of avian influenza, the following measures as indicated in Table 1, Table 2 and Table 3 can be taken;

1. increasing μ_f by killing infected poultry,
2. reduce β_f, β_m and β_h by closing down farms and markets where there is infected poultry to avoid continuous contact or transmission to human,
3. increase α_0, this is achieved by increased number of susceptible poultry that move to market. If the susceptible poultry are higher than the infected poultry in the market, the chances of human having contact with infected infected poultry will reduce.

Table 1: Table showing the impact of μ_f on the reproduction number.

μ_f	0.4500	0.600	0.7500	0.900	1.050	1.200	1.3500	1.500	1.650	1.800
$\Re_{01}$	1.8396	1.5363	1.3189	1.1554	1.0280	0.9259	0.8422	0.7724	0.7133	0.6626
$\Re_{02}$	2.8518	2.2963	1.9219	1.6525	1.4493	1.2906	1.1632	1.0588	0.9715	0.8975

Table 2: Table showing the impact of β_f on the reproduction number.

β_f	0.0190	0.0170	0.0150	0.0130	0.0110	0.0090	0.0070	0.0050	0.0030	0.0010
$\Re_{01}$	1.3922	1.2456	1.0991	0.9526	0.8060	0.6595	0.5129	0.3664	0.2198	0.0733
$\Re_{02}$	0.7688	0.7688	0.7688	0.7688	0.7688	0.7688	0.7688	0.7688	0.7688	0.7688

Table 3: Table showing the impact of β_m on the reproduction number.

β_m	0.0090	0.0081	0.0072	0.0063	0.0054	0.0045	0.0036	0.0027	0.0018	0.0009
$\Re_{01}$	0.4103	0.4103	0.4103	0.4103	0.4103	0.4103	0.4103	0.4103	0.4103	0.4103
$\Re_{02}$	3.8438	3.4594	3.07550	2.6906	2.3063	1.9219	1.5375	1.1531	0.7688	0.3844

6 Conclusion

In this paper, we combined human and poultry to developed an SI-SI-SEIR dynamic model of avian influenza A(H7N9) with the inclusion of re-infection and transmission of the disease occurring both in farm and market. The reproduction number we obtained is sufficient to establish the following;

1. there exist the disease-free equilibrium U^0 which is globally asymptotically stable whenever $\Re_{01} < 1$ and $\Re_{02} < 1$, hence, the disease dies out (see Figure 5.1).

2. there exist the boundary equilibrium U^* which is globally asymptotically stable whenever $\Re_{01} < 1$ and $\Re_{02} > 1$, hence, the disease will remain in the population and finally lead to epidemic (see Figure 5.3).

3. there exist another boundary equilibrium U^{**}, this boundary equilibrium U^{**} is globally asymptotically stable whenever $\Re_{01} > 1$ and $\Re_{02} < 1$, the disease will be sustained in the population and may lead to epidemic (see Figure 5.5).

4. If $\Re_{01} > 1$ and $\Re_{02} > 1$, there exist the endemic equilibrium U^{***}, we showed that the endemic equilibrium U^{***} is globally asymptotically stable and this means that the disease will spread widely in the population (see Figure 5.7).

From the ongoing, it is deduced that the spread of avian influenza can be reduced by taking the following measures;

1. by killing infected poultry,

2. by isolating the infected poultries in the farm from the healthy ones (i.e. reducing β_f),

3. by killing the infected poultries in the market (i.e. reducing β_m).

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